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EFFECTS OF MICRO-STRUCTURE ON THE STRESS CONCENTRATION AT A SPHERICAL CAVITY

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Abstract—In this paper the boundary-value problem for the stress concentration at a spherical cavity in a field of isotropic tension is solved within the framework of Mindlin's theory of an elastic continuum with a deformable micro-structure. It is found that the stress concentration factor is moderately larger than the 3/2 of classical elasticity for a wide range of material properties and ratios of radius of cavity to a length parameter of the material—with a critical ratio, nearly independent of the remaining material properties, for which the stress concentration factor is a maximum.

INTRODUCTION

IN THIS paper a boundary-value problem is solved within the framework of Mindlin's [1] theory of an elastic continuum with a deformable micro-structure. In the first section the equations of the continuum are presented, and in the following section a set of necessary and sufficient conditions on the elastic constants for the potential energy density to be positive definite is displayed.

In the third section the differential equations which govern radially symmetric problems are derived and their general solution is exhibited. A linear differential operator,

$$(1-\lambda_1^2 D_r^2)(1-\lambda_2^2 D_r^2)D_r^2,$$

where

$$D_r^2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{2}{r^2},$$

arises, and it is shown that λ_1^2 and λ_2^2 are positive, real-valued parameters which depend on material properties.

In the fourth section the boundary-value problem for the stress concentration at a spherical cavity in a medium of infinite extent subject to a field of isotropic tension is formulated, and the solution is obtained. The nondimensionalized form of the solution depends upon eight ratios of elastic constants and the ratio of a length parameter of the material to the radius of the cavity. In the fifth section, by utilizing experimental information and analogies with the classical theory of elasticity, ranges of values for these ratios are selected.

Calculations are performed for several values of each ratio, covering a wide range, and the results are presented in the sixth section. In contrast to the solution obtained by employing the classical theory of elasticity, where the result is a constant, 3/2, independent of material properties and the radius of the cavity, the solution obtained by employing the theory of an elastic material with micro-structure shows a stress concentration factor which depends on both material properties and radius. In all the cases considered, the stress concentration factor is higher than the classical 3/2, and there appears to be a critical radius of cavity, for each material, at which the stress concentration factor reaches a maximum. The ratio of the critical radius, to the material property chosen as a reference length, is insensitive to wide variations of the eight remaining material properties.

EQUATIONS OF THE ELASTIC CONTINUUM WITH MICRO-STRUCTURE

In [1], Mindlin derived the following relations in rectangular cartesian coordinates x_i, $i = 1, 2, 3$, for an elastic continuum with a deformable micro-structure: the twelve stress equations of equilibrium (omitting body forces)

$$\begin{aligned} \partial_i(\tau_{ij}+\sigma_{ij}) &= 0, \\ \partial_i\mu_{ijk}+\sigma_{jk} &= 0; \end{aligned} \tag{1}$$

and the twelve traction boundary conditions,

$$\begin{aligned} t_j &= n_i(\tau_{ij}+\sigma_{ij}), \\ T_{jk} &= n_i\mu_{ijk}, \end{aligned} \tag{2}$$

where $\partial_i \equiv \partial/\partial x_i$ is the gradient operator,

$$\tau_{ij} \equiv \frac{\partial W}{\partial \varepsilon_{ij}}, \qquad \sigma_{ij} \equiv \frac{\partial W}{\partial \gamma_{ij}}, \qquad \mu_{ijk} \equiv \frac{\partial W}{\partial \kappa_{ijk}}. \tag{3}$$

In (3), τ_{ij} is the *Cauchy stress*, σ_{ij} is the *relative stress*, μ_{ijk} is the *double stress*, and $W(\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk})$ is the potential energy density with

$$\varepsilon_{ij} \equiv \tfrac{1}{2}(\partial_i u_j+\partial_j u_i), \qquad \gamma_{ij} \equiv \partial_i u_j-\psi_{ij}, \qquad \kappa_{ijk} \equiv \partial_i\psi_{jk}, \tag{4}$$

where ε_{ij} is the *macro-strain*, γ_{ij} is the *relative deformation*, κ_{ijk} is the *micro-deformation gradient*, u_j is the *macro-displacement*, and ψ_{ij} is the *micro-deformation*.

The potential energy density, W, is taken to be a homogeneous quadratic function of the forty-two variables ε_{ij}, γ_{ij}, κ_{ijk}. In the case of a centrosymmetric, isotropic material (referred to as isotropic in the sequel) W reduces to

$$\begin{aligned} W = {} & \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+\tfrac{1}{2}b_1\gamma_{ii}\gamma_{jj}+\tfrac{1}{2}b_2\gamma_{ij}\gamma_{ij} \\ & +\tfrac{1}{2}b_3\gamma_{ij}\gamma_{ji}+g_1\gamma_{ii}\varepsilon_{jj}+g_2(\gamma_{ij}+\gamma_{ji})\varepsilon_{ij} \\ & +a_1\kappa_{iik}\kappa_{kjj}+a_2\kappa_{iik}\kappa_{jkj}+\tfrac{1}{2}a_3\kappa_{iik}\kappa_{jjk} \\ & +\tfrac{1}{2}a_4\kappa_{ijj}\kappa_{ikk}+a_5\kappa_{ijj}\kappa_{kik}+\tfrac{1}{2}a_8\kappa_{iji}\kappa_{kjk} \\ & +\tfrac{1}{2}a_{10}\kappa_{ijk}\kappa_{ijk}+a_{11}\kappa_{ijk}\kappa_{jki}+\tfrac{1}{2}a_{13}\kappa_{ijk}\kappa_{ikj} \\ & +\tfrac{1}{2}a_{14}\kappa_{ijk}\kappa_{jik}+\tfrac{1}{2}a_{15}\kappa_{ijk}\kappa_{kji}, \end{aligned} \tag{5}$$

and the constitutive equations (3) become

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk}+2\mu\varepsilon_{ij}+g_1\delta_{ij}\gamma_{kk}+g_2(\gamma_{ij}+\gamma_{ji}), \tag{6a}$$

$$\sigma_{ij} = g_1\delta_{ij}\varepsilon_{kk}+2g_2\varepsilon_{ij}+b_1\delta_{ij}\gamma_{kk}+b_2\gamma_{ij}+b_3\gamma_{ji}, \tag{6b}$$

$$\begin{aligned}\mu_{ijk} = {} & a_1(\kappa_{ppi}\delta_{jk}+\kappa_{kpp}\delta_{ij})+a_2(\kappa_{ppj}\delta_{ik}+\kappa_{pkp}\delta_{ij})\\ & +a_3\kappa_{ppk}\delta_{ij}+a_4\kappa_{ipp}\delta_{jk}+a_5(\kappa_{jpp}\delta_{ik}+\kappa_{pip}\delta_{jk})\\ & +a_8\kappa_{pjp}\delta_{ik}+a_{10}\kappa_{ijk}+a_{11}(\kappa_{kij}+\kappa_{jki})\\ & +a_{13}\kappa_{ikj}+a_{14}\kappa_{jik}+a_{15}\kappa_{kji},\end{aligned} \tag{6c}$$

where δ_{ij} is the Kronecker symbol.

Inserting the definitions (4) into the constitutive equations (6) and the latter into the stress equations of equilibrium (1), twelve equations on the twelve variables u_i, ψ_{ij} are obtained which, in invariant form, may be written

$$\begin{aligned}&(\mu+2g_2+b_2)\nabla^2\mathbf{u}+(\lambda+\mu+2g_1+2g_2+b_1+b_3)\nabla\nabla\cdot\mathbf{u}\\ &\quad -(g_1+b_1)\nabla(\mathbf{I}:\boldsymbol{\psi})-(g_2+b_2)\nabla\cdot\boldsymbol{\psi}-(g_2+b_3)\boldsymbol{\psi}\cdot\nabla = \mathbf{0},\\ &(a_1+a_5)[\mathbf{I}\nabla\cdot\boldsymbol{\psi}\cdot\nabla+\nabla\nabla(\mathbf{I}:\boldsymbol{\psi})]+(a_2+a_{11})(\nabla\cdot\boldsymbol{\psi}\nabla+\nabla\boldsymbol{\psi}\cdot\nabla)\\ &\quad +(a_3+a_{14})\nabla\nabla\cdot\boldsymbol{\psi}+a_4\mathbf{I}\nabla^2(\mathbf{I}:\boldsymbol{\psi})+(a_8+a_{15})\boldsymbol{\psi}\cdot\nabla\nabla\\ &\quad +a_{10}\nabla^2\boldsymbol{\psi}+a_{13}\nabla^2\boldsymbol{\psi}_c+g_1\mathbf{I}\nabla\cdot\mathbf{u}+g_2(\nabla\mathbf{u}+\mathbf{u}\nabla)\\ &\quad +b_1\mathbf{I}(\nabla\cdot\mathbf{u}-\mathbf{I}:\boldsymbol{\psi})+b_2(\nabla\mathbf{u}-\boldsymbol{\psi})+b_3(\mathbf{u}\nabla-\boldsymbol{\psi}_c) = \mathbf{0},\end{aligned} \tag{7}$$

where $\mathbf{I}$ is the idemfactor, $\boldsymbol{\psi}_c$ is the conjugate of $\boldsymbol{\psi}$, and ∇ is the gradient operator.

POSITIVE DEFINITENESS OF THE POTENTIAL ENERGY DENSITY

The quadratic form (5) can be written in matrix notation as

$$W = \tfrac{1}{2}\mathbf{X}\mathbf{A}\mathbf{X}^T \tag{8}$$

where $\mathbf{A}$ is the symmetric matrix,

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{63} & \mathbf{0}_{63} & \mathbf{0}_{63} & \mathbf{0}_{67} & \mathbf{0}_{67} & \mathbf{0}_{67} & \mathbf{0}_{66}\\ \mathbf{0}_{36} & \mathbf{A}_2 & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{36}\\ \mathbf{0}_{36} & \mathbf{0}_{33} & \mathbf{A}_2 & \mathbf{0}_{33} & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{36}\\ \mathbf{0}_{36} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{A}_2 & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{37} & \mathbf{0}_{36}\\ \mathbf{0}_{76} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{A}_3 & \mathbf{0}_{77} & \mathbf{0}_{77} & \mathbf{0}_{76}\\ \mathbf{0}_{76} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{77} & \mathbf{A}_3 & \mathbf{0}_{77} & \mathbf{0}_{76}\\ \mathbf{0}_{76} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{77} & \mathbf{0}_{77} & \mathbf{A}_3 & \mathbf{0}_{76}\\ \mathbf{0}_{76} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{73} & \mathbf{0}_{77} & \mathbf{0}_{77} & \mathbf{0}_{77} & \mathbf{A}_4 \end{bmatrix}, \tag{9}$$

with

$$\mathbf{A}_1 \equiv \begin{bmatrix} b_1+b_2+b_3 & b_1 & b_1 & g_1+2g_2 & g_1 & g_1 \\ b_1 & b_1+b_2+b_3 & b_1 & g_1 & g_1+2g_2 & g_1 \\ b_1 & b_1 & b_1+b_2+b_3 & g_1 & g_1 & g_1+2g_2 \\ g_1+2g_2 & g_1 & g_1 & \lambda+2\mu & \lambda & \lambda \\ g_1 & g_1+2g_2 & g_1 & \lambda & \lambda+2\mu & \lambda \\ g_1 & g_1 & g_1+2g_2 & \lambda & \lambda & \lambda+2\mu \end{bmatrix},$$

$$\mathbf{A}_2 \equiv \begin{bmatrix} b_2 & b_3 & g_2 \\ b_3 & b_2 & g_2 \\ g_2 & g_2 & \mu \end{bmatrix}, \tag{10}$$

$$\mathbf{A}_3 \equiv \begin{bmatrix} \xi_1 & \xi_2 & \xi_2 & a_2+a_5+a_8 & a_2+a_5+a_8 & a_1+a_2+a_3 & a_1+a_2+a_3 \\ \xi_2 & \xi_3-a_4 & a_4 & a_5+a_{11}+a_{14} & a_5 & a_1+a_{11}+a_{15} & a_1 \\ \xi_2 & a_4 & \xi_3-a_4 & a_5 & a_5+a_{11}+a_{14} & a_1 & a_1+a_{11}+a_{15} \\ a_2+a_5+a_8 & a_5+a_{11}+a_{14} & a_5 & a_8+a_{10}+a_{15} & a_8 & a_2+a_{11}+a_{13} & a_2 \\ a_2+a_5+a_8 & a_5 & a_5+a_{11}+a_{14} & a_8 & a_8+a_{10}+a_{15} & a_2 & a_2+a_{11}+a_{13} \\ a_1+a_2+a_3 & a_1+a_{11}+a_{15} & a_1 & a_2+a_{11}+a_{13} & a_2 & a_3+a_{10}+a_{14} & a_3 \\ a_1+a_2+a_3 & a_1 & a_1+a_{11}+a_{15} & a_2 & a_2+a_{11}+a_{13} & a_3 & a_3+a_{10}+a_{14} \end{bmatrix}$$

$$\mathbf{A}_4 \equiv \begin{bmatrix} a_{10} & a_{11} & a_{11} & a_{14} & a_{15} & a_{13} \\ a_{11} & a_{10} & a_{11} & a_{13} & a_{14} & a_{15} \\ a_{11} & a_{11} & a_{10} & a_{15} & a_{13} & a_{14} \\ a_{14} & a_{13} & a_{15} & a_{10} & a_{11} & a_{11} \\ a_{15} & a_{14} & a_{13} & a_{11} & a_{10} & a_{11} \\ a_{13} & a_{15} & a_{14} & a_{11} & a_{11} & a_{10} \end{bmatrix}$$

where,

$$\xi_1 \equiv 2a_1+2a_2+a_3+a_4+2a_5+a_8+a_{10}+2a_{11}+a_{13}+a_{14}+a_{15},$$
$$\xi_2 \equiv a_1+a_4+a_5, \tag{11}$$
$$\xi_3 \equiv 2a_4+a_{10}+a_{13}.$$

The matrix $\mathbf{0}_{mn}$ is the zero matrix with m rows and n columns, $\mathbf{X}$ is the row vector,

$$\mathbf{X} \equiv [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_8],$$

with

$$\mathbf{X}_1 \equiv [\gamma_{33}, \gamma_{22}, \gamma_{11}, \varepsilon_{33}, \varepsilon_{22}, \varepsilon_{11}],$$
$$\mathbf{X}_2 \equiv [\gamma_{32}, \gamma_{23}, 2\varepsilon_{23}],$$
$$\mathbf{X}_3 \equiv [\gamma_{13}, \gamma_{31}, 2\varepsilon_{31}],$$
$$\mathbf{X}_4 \equiv [\gamma_{21}, \gamma_{12}, 2\varepsilon_{12}],$$
$$\mathbf{X}_5 \equiv [\kappa_{111}, \kappa_{122}, \kappa_{133}, \kappa_{212}, \kappa_{313}, \kappa_{221}, \kappa_{331}],$$
$$\mathbf{X}_6 \equiv [\kappa_{222}, \kappa_{233}, \kappa_{211}, \kappa_{323}, \kappa_{121}, \kappa_{332}, \kappa_{112}],$$
$$\mathbf{X}_7 \equiv [\kappa_{333}, \kappa_{311}, \kappa_{322}, \kappa_{131}, \kappa_{232}, \kappa_{113}, \kappa_{223}],$$
$$\mathbf{X}_8 \equiv [\kappa_{123}, \kappa_{231}, \kappa_{312}, \kappa_{213}, \kappa_{321}, \kappa_{132}],$$

and $\mathbf{X}^T$ is the transpose of $\mathbf{X}$.

A well known theorem of algebra (see for example [2]) states that, if $\mathbf{M}$ is a real, nxn, symmetric matrix and if $\mathbf{Y}$ is a real row vector with n elements, then a set of necessary and sufficient conditions for the quadratic form, $q \equiv \mathbf{YMY}^T$, to be positive definite can be obtained by requiring the discriminants of the quadratic forms obtained from q by eliminating, successively, $(n-1)$, $(n-2)$, $\ldots$, 0 of the variables, to be greater than zero.

In the present case, the matrix $\mathbf{A}$ of the quadratic form (8) has a higher degree of symmetry than that required by the theorem, and the number of inequalities needed to insure positive definiteness of the potential energy density is less than the number of variables. The step-by-step evaluation of the discriminants can be performed (see Appendix), and equivalent sets of necessary and sufficient conditions for positive definiteness can be selected. One such set is the following:

$$\tilde{\mu} \equiv \mu - \frac{2g_2^2}{b_2+b_3} > 0,$$

$$3\tilde{\lambda}+2\tilde{\mu} \equiv 3\lambda+2\mu - \frac{(3g_1+2g_2)^2}{3b_1+b_2+b_3} > 0,$$

$$b_2+b_3 > 0,$$

$$b_2-b_3 > 0,$$

$$3b_1+b_2+b_3 > 0,$$

$$a_{10}+a_{13} > 0,$$

$$a_{10}-a_{11} > 0,$$

$$
\begin{aligned}
&\xi_1 > 0,\\
&\xi_4 \equiv a_{10}+2a_{11}+a_{13}+a_{14}+a_{15} > 0,\\
&\xi_5 \equiv a_{10}+2a_{11}-a_{13}-a_{14}-a_{15} > 0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\Delta_1 \equiv \xi_1\xi_3-2\xi_2^2 > 0,\\
&\Delta_2 \equiv (a_{10}+a_{13})(a_{10}+a_{15})-(a_{11}+a_{14})^2 > 0;
\end{aligned}
$$

$$\Delta_3 \equiv (a_{10}-a_{11})^2-(a_{13}-a_{15})^2-(a_{14}-a_{15})^2+(a_{13}-a_{15})(a_{14}-a_{15}) > 0,$$

$$\Delta_4 \equiv \begin{vmatrix} \frac{1}{2}\xi_1 & \xi_2 & a_2+a_5+a_8 \\ \xi_2 & \xi_3 & 2a_5+a_{11}+a_{14} \\ a_2+a_5+a_8 & 2a_5+a_{11}+a_{14} & 2a_8+a_{10}+a_{15} \end{vmatrix} > 0,$$

$$\Delta_5 \equiv \begin{vmatrix} \frac{1}{2}\xi_1 & \xi_2 & a_2+a_5+a_8 & a_1+a_2+a_3 \\ \xi_2 & \xi_3 & 2a_5+a_{11}+a_{14} & 2a_1+a_{11}+a_{15} \\ a_2+a_5+a_8 & 2a_5+a_{11}+a_{14} & 2a_8+a_{10}+a_{15} & 2a_2+a_{11}+a_{13} \\ a_1+a_2+a_3 & 2a_1+a_{11}+a_{15} & 2a_2+a_{11}+a_{13} & 2a_3+a_{10}+a_{14} \end{vmatrix} > 0.$$

RADIAL SYMMETRY

Governing equations

For radially symmetric problems we take

$$
\begin{aligned}
\mathbf{u} &= u_r(r)\mathbf{e}_r,\\
\boldsymbol{\psi} &= \psi_{rr}(r)\mathbf{e}_r\mathbf{e}_r+\psi_{\theta\theta}(r)\mathbf{e}_\theta\mathbf{e}_\theta+\psi_{\theta\theta}(r)\mathbf{e}_\phi\mathbf{e}_\phi,
\end{aligned}
\tag{13}
$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are the unit vectors corresponding to the spherical coordinates r, θ, ϕ. Inserting (13) into (7) and taking linear combinations of the resulting equations we obtain the system:

$$k_{11}D_0D_2u_r-k_{12}D_3\psi_{rr}^D-k_{13}D_0\psi_s = 0, \tag{14a}$$

$$k_{21}D_1u_r+(k_{22}D_1D_3-k'_{22})\psi_{rr}^D+k_{23}D_1D_0\psi_s = 0, \tag{14b}$$

$$k_{31}D_2u_r+k_{32}D_2D_3\psi_{rr}^D+(k_{33}D_2D_0-k'_{33})\psi_s = 0, \tag{14c}$$

where,

$$
\begin{aligned}
\psi_s &\equiv \tfrac{1}{3}(\mathbf{I}:\boldsymbol{\psi}) = \tfrac{1}{3}(\psi_{rr}+2\psi_{\theta\theta}),\\
\psi_{rr}^D &\equiv \mathbf{e}_r\mathbf{e}_r:(\boldsymbol{\psi}-\psi_s\mathbf{I}) = \tfrac{2}{3}(\psi_{rr}-\psi_{\theta\theta});
\end{aligned}
\tag{15}
$$

the linear differential operators D_i are defined by

$$D_0 \equiv \frac{\mathrm{d}}{\mathrm{d}r}, \qquad D_1 \equiv \frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r}, \qquad D_2 \equiv \frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}, \qquad D_3 \equiv \frac{\mathrm{d}}{\mathrm{d}r}+\frac{3}{r},$$

and

$$
\begin{aligned}
k_{11} &= \lambda+2\mu+2g_1+4g_2+b_1+b_2+b_3,\\
k_{22} &= \xi_1-2\xi_2+\tfrac{1}{2}\xi_3,\\
k_{33} &= \xi_1+4\xi_2+2\xi_3,\\
k_{23} &= k_{32} = \xi_1+\xi_2-\xi_3,\\
k_{31} &= k_{13} = 3g_1+2g_2+3b_1+b_2+b_3,\\
k_{12} &= k_{21} = 2g_2+b_2+b_3,\\
k'_{22} &= \tfrac{3}{2}(b_2+b_3),\\
k'_{33} &= 3(3b_1+b_2+b_3).
\end{aligned} \tag{16}
$$

The form of equations (14) is suggested by the form of equations (8.4) of [1], and the definitions of the k_{ij} are the same as in [1]. Further motivation is provided by the form of Mindlin's equations (7) of [3]. Employing the assumptions (13) and the definitions (15) the relations (4) become:

$$
\begin{aligned}
&\varepsilon_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \qquad \gamma_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}-\psi^D_{rr}-\psi_s,\\
&\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u_r}{r}, \qquad \gamma_{\theta\theta} = \gamma_{\phi\phi} = \frac{u_r}{r}-\psi_s+\tfrac{1}{2}\psi^D_{rr},\\
&\varepsilon_{\alpha\beta} = 0, \quad \alpha\neq\beta; \qquad \gamma_{\alpha\beta} = 0, \quad \alpha\neq\beta;\\
&\kappa_{rrr} = \frac{\mathrm{d}\psi^D_{rr}}{\mathrm{d}r}+\frac{\mathrm{d}\psi_s}{\mathrm{d}r},\\
&\kappa_{r\theta\theta} = \kappa_{r\phi\phi} = -\tfrac{1}{2}\frac{\mathrm{d}\psi^D_{rr}}{\mathrm{d}r}+\frac{\mathrm{d}\psi_s}{\mathrm{d}r},\\
&\kappa_{\theta\theta r} = \kappa_{\theta r\theta} = \kappa_{\phi\phi r} = \kappa_{\phi r\phi} = \frac{3}{2}\frac{\psi^D_{rr}}{r},
\end{aligned} \tag{17}
$$

and all other $\kappa_{\alpha\beta\gamma} = 0$, where the indices α, β, γ range over the values r, θ, ϕ.

Inserting (17) into the constitutive equations (6), we obtain

$$
\begin{aligned}
\tau_{rr} &= (\lambda+2\mu+g_1+2g_2)\frac{\mathrm{d}u_r}{\mathrm{d}r}+2(\lambda+g_1)\frac{u_r}{r}-(3g_1+2g_2)\psi_s-2g_2\psi^D_{rr},\\
\tau_{\theta\theta} &= \tau_{\phi\phi} = (\lambda+g_1)\frac{\mathrm{d}u_r}{\mathrm{d}r}+2(\lambda+\mu+g_1+g_2)\frac{u_r}{r}-(3g_1+2g_2)\psi_s+g_2\psi^D_{rr},\\
\tau_{\alpha\beta} &= 0, \qquad \alpha\neq\beta;\\
\sigma_{rr} &= (g_1+2g_2+b_1+b_2+b_3)\frac{\mathrm{d}u_r}{\mathrm{d}r}+2(b_1+g_1)\frac{u_r}{r}-(3b_1+b_2+b_3)\psi_s-(b_2+b_3)\psi^D_{rr},\\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = (g_1+b_1)\frac{\mathrm{d}u_r}{\mathrm{d}r}+(2g_1+2g_2+2b_1+b_2+b_3)\frac{u_r}{r}-(3b_1+b_2+b_3)\psi_s+\tfrac{1}{2}(b_2+b_3)\psi^D_{rr},\\
\sigma_{\alpha\beta} &= 0, \qquad \alpha\neq\beta; \qquad\qquad (18)\\
\mu_{rrr} &= [\xi_1-\xi_2]\frac{\mathrm{d}\psi^D_{rr}}{\mathrm{d}r}+3[(a_1+a_2+a_3)+(a_2+a_5+a_8)]\frac{\psi^D_{rr}}{r}+[\xi_1+2\xi_2]\frac{\mathrm{d}\psi_s}{\mathrm{d}r},
\end{aligned}
$$

$$\mu_{r\theta\theta} = \mu_{r\phi\phi} = [\xi_2 - \tfrac{1}{2}\xi_3]\frac{d\psi_{rr}^D}{dr} + \tfrac{3}{2}[(2a_1 + a_{11} + a_{15}) + (2a_5 + a_{11} + a_{14})]\frac{\psi_{rr}^D}{r} + [\xi_2 + \xi_3]\frac{d\psi_s}{dr},$$

$$\mu_{\theta\theta r} = \mu_{\phi\phi r} = [(a_1 + a_2 + a_3) - \tfrac{1}{2}(2a_1 + a_{11} + a_{15})]\frac{d\psi_{rr}^D}{dr} + \tfrac{3}{2}[(2a_2 + a_{11} + a_{13}) + (2a_3 + a_{10} + a_{14})]\frac{\psi_{rr}^D}{r} + [(a_1 + a_2 + a_3) + (2a_1 + a_{11} + a_{15})]\frac{d\psi_s}{dr},$$

$$\mu_{\theta r\theta} = \mu_{\phi r\phi} = [(a_2 + a_5 + a_8) - \tfrac{1}{2}(2a_5 + a_{11} + a_{14})]\frac{d\psi_{rr}^D}{dr} + \tfrac{3}{2}[(2a_2 + a_{11} + a_{13}) + (2a_8 + a_{10} + a_{15})]\frac{\psi_{rr}^D}{r} + [(a_2 + a_5 + a_8) + (2a_5 + a_{11} + a_{14})]\frac{d\psi_s}{dr},$$

and all other $\mu_{\alpha\beta\gamma} = 0$.

General solution

The system of equations (14), with variable coefficients, may be transformed to one, with constant coefficients, in the variables

$$\Theta_1 = u_r, \qquad \Theta_2 = D_3\psi_{rr}^D, \qquad \Theta_3 = D_0\psi_s \tag{19}$$

by the operations

$$D_3(14b), \qquad k_{12}(14a) - k_{11}D_3(14b), \qquad k_{13}(14a) - k_{11}D_0(14c).$$

Noting that

$$D_3D_1 = D_0D_2 = \frac{d^2}{dr^2} + \frac{2}{r^2}\frac{d}{dr} - \frac{2}{r^2} \equiv D_r^2,$$

we obtain

$$\begin{aligned} k_{21}D_r^2\Theta_1 + [k_{22}D_r^2 - k'_{22}]\Theta_2 + k_{23}D_r^2\Theta_3 &= 0, \\ [k_{11}k_{22}D_r^2 - (k_{11}k'_{22} - k_{12}^2)]\Theta_2 + [k_{11}k_{23}D_r^2 + k_{12}k_{13}]\Theta_3 &= 0, \\ [k_{11}k_{23}D_r^2 + k_{12}k_{13}]\Theta_2 + [k_{11}k_{33}D_r^2 - (k_{11}k'_{33} - k_{13}^2)]\Theta_3 &= 0. \end{aligned} \tag{20}$$

Now the Θ_i, $i = 1, 2, 3$, satisfy the equation

$$(1 - \lambda_1^2 D_r^2)(1 - \lambda_2^2 D_r^2)D_r^2\Theta_i = 0 \tag{21}$$

where

$$\lambda_1^2, \lambda_2^2 = [b \pm (b^2 - 4ac)^{\frac{1}{2}}]/2c,$$

and

$$\begin{aligned} a &= k_{11}(k_{22}k_{33}-k_{23}^2), \\ b &= k_{11}(k'_{22}k_{33}+k_{22}k'_{33})+2k_{12}k_{13}k_{23}-k_{22}k_{13}^2-k_{33}k_{12}^2, \\ c &= k_{11}k'_{22}k'_{33}-k_{12}^2k'_{33}-k_{13}^2k'_{22}. \end{aligned} \tag{22}$$

The nature of the solutions of (21) depends on the character of the λ_i^2. In the Appendix, the conditions for positive definiteness are employed to prove that

$$a > 0, \qquad b > 0, \qquad c > 0, \qquad b^2-4ac \geq 0, \tag{23}$$

and hence that the λ_i^2 are real and positive.

The operators in (21) are commutative, and hence the general solution for each Θ_i can be written as the sum

$$\Theta_i = \Theta_i^0+\Theta_i^1+\Theta_i^2, \qquad i = 1, 2, 3, \tag{24}$$

where Θ_i^0, Θ_i^1, Θ_i^2 are the solutions of

$$D_r^2\Theta_i^0 = 0, \qquad (1-\lambda_1^2D_r^2)\Theta_i^1 = 0, \qquad (1-\lambda_2^2D_r^2)\Theta_i^2 = 0. \tag{25}$$

The general solutions of equations (25) are:

$$\Theta_i^0 = c_{i1}r+c_{i2}r^{-2}, \tag{26a}$$

$$\Theta_i^1 = c_{i3}e^{r/\lambda_1}[\lambda_1^2r^{-2}-\lambda_1r^{-1}]+c_{i4}e^{-r/\lambda_1}[\lambda_1^2r^{-2}+\lambda_1r^{-1}], \tag{26b}$$

$$\Theta_i^2 = c_{i5}e^{r/\lambda_2}[\lambda_2^2r^{-2}-\lambda_2r^{-1}]+c_{i6}e^{-r/\lambda_2}[\lambda_2^2r^{-2}+\lambda_2r^{-1}], \tag{26c}$$

where the c_{ij} are constants. The solution for the variables u_r, ψ_{rr}^D and ψ_s is now a straightforward matter. The solutions (26) are inserted in (24) and the latter then substituted into (20) to determine the relations among the constants c_{ij}. Once this has been done, the second and third of equations (19) serve as simple differential equations on the variables ψ_{rr}^D and ψ_s. These equations are solved and two additional constants of integration are introduced. The expressions for u_r, ψ_{rr}^D and ψ_s are then substituted into equations (14) to find the remaining relations among the constants. The result is the following general solution of equations (14):

$$\begin{aligned} u_r &= c_1r+c_2r^{-2}+\beta_1(c_3f_{11}e^{r/\lambda_1}+c_4f_{10}e^{-r/\lambda_1}) \\ &\quad +\beta_2(c_5f_{21}e^{r/\lambda_2}+c_6f_{20}e^{-r/\lambda_2}), \\ \psi_{rr}^D &= c_2\zeta_2r^{-3}-\alpha_1r(c_3h_{11}e^{r/\lambda_1}+c_4h_{10}e^{-r/\lambda_1}) \\ &\quad -\alpha_2r(c_5h_{21}e^{r/\lambda_2}+c_6h_{20}e^{-r/\lambda_2}), \\ \psi_s &= c_1\zeta_1-\lambda_1^2r^{-1}(c_3e^{r/\lambda_1}+c_4e^{-r/\lambda_1})-\lambda_2^2r^{-1}(c_5e^{r/\lambda_2}+c_6e^{-r/\lambda_2}), \end{aligned} \tag{27}$$

where, for $i = 1, 2$, and $j = 0, 1$,

$$\begin{aligned} f_{ij} &= (\lambda_i/r)^2+(-1)^j(\lambda_i/r), \\ h_{ij} &= 3(\lambda_i/r)^4+3(-1)^j(\lambda_i/r)^3+(\lambda_i/r)^2, \end{aligned}$$

the c's are arbitrary constants and

$$\alpha_i = -\frac{[k_{11}k_{33}-\lambda_i^2(k_{11}k'_{33}-k_{13}^2)]}{[k_{11}k_{23}+\lambda_i^2k_{12}k_{13}]} = -\frac{[k_{11}k_{23}+\lambda_i^2k_{12}k_{13}]}{[k_{11}k_{22}-\lambda_i^2(k_{11}k'_{22}-k_{12}^2)]},$$

$$\beta_i = -\frac{[k_{23}+\alpha_i(k_{22}-\lambda_i^2k'_{22})]}{k_{12}} = -\frac{[(k_{33}-\lambda_i^2k'_{33})+\alpha_ik_{23}]}{k_{13}} = \frac{[\alpha_ik_{12}+k_{13}]\lambda_i^2}{k_{11}}, \tag{28}$$

$$\zeta_1 = 3k_{13}/k'_{33}, \qquad \zeta_2 = -3k_{12}/k'_{22}.$$

STRESS CONCENTRATION AT A SPHERICAL CAVITY

The solution of the boundary-value problem for a spherical cavity of radius, a, in a medium of infinite extent subject to an isotropic tension (t_0 per unit area) at infinity is obtained by applying the following boundary conditions to the general solution:

$$\text{on } r = a: \tau_{rr}+\sigma_{rr} = 0; \qquad \mu_{rrr} = 0; \qquad \mu_{r\theta\theta} = \mu_{r\phi\phi} = 0; \tag{29}$$

$$\text{as } r \to \infty: \tau_{rr}+\sigma_{rr} \to t_0; \qquad \mu_{rrr}, \quad \mu_{r\theta\theta}, \quad \mu_{r\phi\phi} \to 0. \tag{30}$$

Inserting (27) into (18) and employing (28), we find after some manipulation,

$$\begin{aligned}\tau_{rr}+\sigma_{rr} = {} & c_1(3\tilde{\lambda}+2\tilde{\mu})-4c_2\tilde{\mu}r^{-3}+\Gamma_1r^{-1}(c_3f_{11}e^{r/\lambda_1}+c_4f_{10}e^{-r/\lambda_1})\\ & +\Gamma_2r^{-1}(c_5f_{21}e^{r/\lambda_2}+c_6f_{20}e^{-r/\lambda_2}),\end{aligned} \tag{31}$$

$$\begin{aligned}\mu_{rrr} = {} & -3c_2\zeta_2\xi_4r^{-4}+c_3e^{r/\lambda_1}(3\xi_4\alpha_1h_{11}+\chi_1f_{11})\\ & +c_4e^{-r/\lambda_1}(3\xi_4\alpha_1h_{10}+\chi_1f_{10})\\ & +c_5e^{r/\lambda_2}(3\xi_4\alpha_2h_{21}+\chi_2f_{21})+c_6e^{-r/\lambda_2}(3\xi_4\alpha_2h_{20}+\chi_2f_{20}),\end{aligned} \tag{32}$$

$$\begin{aligned}2\mu_{r\theta\theta} = 2\mu_{r\phi\phi} = {} & 3c_2\zeta_2\xi_4r^{-4}+c_3e^{r/\lambda_1}(-3\xi_4\alpha_1h_{11}+\eta_1f_{11})\\ & +c_4e^{-r/\lambda_1}(-3\xi_4\alpha_1h_{10}+\eta_1f_{10})+c_5e^{r/\lambda_2}(-3\xi_4\alpha_2h_{21}+\eta_2f_{21})\\ & +c_6e^{-r/\lambda_2}(-3\xi_4\alpha_2h_{20}+\eta_2f_{20}),\end{aligned} \tag{33}$$

where

$$\begin{aligned}\Gamma_i &= \beta_i(3\tilde{\lambda}+2\tilde{\mu})-\zeta_1(k_{23}\alpha_i+k_{33}),\\ \chi_i &= \xi_1(1+\alpha_i)+\xi_2(2-\alpha_i),\\ \eta_i &= 2\xi_2(1+\alpha_i)+\xi_3(2-\alpha_i),\end{aligned}$$

for $i = 1, 2$ and $\tilde{\lambda}$, $\tilde{\mu}$, ξ_1, ξ_2, ξ_3, ξ_4, α_i, β_i, ζ_i are defined by (11), (12), and (28). Applying the second of the conditions (30) to equations (32) and (33) we conclude

$$c_3 = c_5 = 0. \tag{34}$$

Applying the first of the conditions (30) to equation (31) we obtain

$$c_1 = t_0/(3\tilde{\lambda}+2\tilde{\mu}). \tag{35}$$

Now, applying the conditions (29) to the equations (31), (32), (33), and employing (34) and (35) we get the following set of simultaneous equations for the determination of the constants c_2, c_4, c_6:

$$
\begin{aligned}
4c_2\tilde{\mu}a^{-3} - c_4 e^{-a/\lambda_1}\Gamma_1 a^{-1} f^a_{10} - c_6 e^{-a/\lambda_2}\Gamma_2 a^{-1} f^a_{20} &= t_0, \\
3c_2\zeta_2\xi_4 a^{-4} - c_4 e^{-a/\lambda_1}(3\xi_4\alpha_1 h^a_{10} + \chi_1 f^a_{10}) - c_6 e^{-a/\lambda_2}(3\xi_4\alpha_2 h^a_{20} + \chi_2 f^a_{20}) &= 0, \\
3c_2\zeta_2\xi_4 a^{-4} - c_4 e^{-a/\lambda_1}(3\xi_4\alpha_1 h^a_{10} - \eta_1 f^a_{10}) - c_6 e^{-a/\lambda_2}(3\xi_4\alpha_2 h^a_{20} - \eta_2 f^a_{20}) &= 0,
\end{aligned}
\tag{36}
$$

where

$$
f^a_{ij} = f_{ij}(\lambda_i/a) \quad \text{and} \quad h^a_{ij} = h_{ij}(\lambda_i/a).
$$

The solution of equations (36) is given by

$$
\begin{aligned}
c_2 &= \frac{t_0 a^3}{4\tilde{\mu}}\left[\frac{1}{1+K}\right], \\
c_4 &= \frac{t_0 a e^{a/\lambda_1}}{f^a_{10}} \frac{[\chi_2+\eta_2]}{[\Gamma_2(\chi_1+\eta_1) - \Gamma_1(\chi_2+\eta_2)]}\left[\frac{K}{1+K}\right], \\
c_6 &= -\frac{t_0 a e^{a/\lambda_2}}{f^a_{20}} \frac{[\chi_1+\eta_1]}{[\Gamma_2(\chi_1+\eta_1) - \Gamma_1(\chi_2+\eta_2)]}\left[\frac{K}{1+K}\right],
\end{aligned}
\tag{37}
$$

where

$$
K = \frac{-3\zeta_2\xi_4 f^a_{10} f^a_{20}[\Gamma_2(\chi_1+\eta_1) - \Gamma_1(\chi_2+\eta_2)]}{4\tilde{\mu}a^2[(3\alpha_2\xi_4 h^a_{20} + \chi_2 f^a_{20})(\chi_1+\eta_1) f^a_{10} - (3\alpha_1\xi_4 h^a_{10} + \chi_1 f^a_{10})(\chi_2+\eta_2) f^a_{20}]}. \tag{38}
$$

We define a stress concentration factor,

$$
\left.\frac{\tau_{\theta\theta}+\sigma_{\theta\theta}}{t_0}\right]_{r=a},
$$

which is the ratio of the force per unit area, across a meridional plane at $r = a$, to the radial force per unit area far from the cavity. Inserting (27) into (18) and employing (34) we find

$$
\begin{aligned}
\tau_{\theta\theta}+\sigma_{\theta\theta} = {} & c_1(3\tilde{\lambda}+2\tilde{\mu}) + 2c_2\tilde{\mu}r^{-3} \\
& -\tfrac{1}{2}c_4 e^{-r/\lambda_1}\Gamma_1\lambda_1^{-2} r[h_{10} - 2(\lambda_1/r)^2 f_{10}], \\
& -\tfrac{1}{2}c_6 e^{-r/\lambda_2}\Gamma_2\lambda_2^{-2} r[h_{20} - 2(\lambda_2/r)^2 f_{20}],
\end{aligned}
\tag{39}
$$

and hence, employing (35) and (37) we find

$$
\left.\frac{\tau_{\theta\theta}+\sigma_{\theta\theta}}{t_0}\right]_{r=a} = \frac{3}{2}\left[1 + \frac{KK'}{3(1+K)}\right], \tag{40}
$$

where

$$
K' = \frac{[f^a_{20}\Gamma_1(\chi_2+\eta_2) - f^a_{10}\Gamma_2(\chi_1+\eta_1)]}{f^a_{10} f^a_{20}[\Gamma_1(\chi_2+\eta_2) - \Gamma_2(\chi_1+\eta_1)]},
$$

and K is defined by (38).

PREPARATION FOR NUMERICAL CALCULATIONS

For the purpose of obtaining numerical values for the stress concentration factor, it is desirable to express the material constants, that appear in K and K', in terms of dimensionless quantities for which estimates of magnitudes can be made. It can be shown that K and K' can be expressed in terms of the following independent ratios of material constants:

$$
\begin{aligned}
\nu_1 &\equiv 2g_2/(b_2+b_3) & \nu_5 &\equiv \tilde{\lambda}/2(\tilde{\lambda}+\tilde{\mu}),\\
\nu_2 &\equiv (3g_1+2g_2)/(3b_1+b_2+b_3), & \nu_6 &\equiv \xi_3/\xi_1,\\
\nu_3 &\equiv (\tilde{\lambda}+2\tilde{\mu})/(b_2+b_3), & \nu_7 &\equiv \xi_2/\xi_1,\\
\nu_4 &\equiv (\tilde{\lambda}+2\tilde{\mu})/(3b_1+b_2+b_3), & \nu_8 &\equiv \xi_4/\xi_1,
\end{aligned}
$$

and an independent ratio of a parameter of the material with the dimension of length, e.g. λ_1 or λ_2, to the radius of the cavity.

The selection of appropriate values for the ratios $\nu_i (i = 1, 2, \ldots, 8)$ would be a simple matter if the elastic constants were known from experiments; but such is not the case. However, by studying Mindlin's [1] solutions for micro-vibrations and plane waves and by solving three simple problems of homogeneous deformation, viz., simple tension, hydrostatic pressure, and shear, an understanding of the physical significances of the ratios ν_i can be gained. Coupling this with the results of neutron scattering experiments and the conditions for positive definiteness of the potential energy density we can obtain what appear to be reasonable ranges of values for these ratios.

Solutions of the problems of simple tension, hydrostatic pressure, and shear are easily obtained by noting that the assumptions

$$\tau_{ij} = \text{constant}, \qquad \sigma_{ij} = 0, \qquad \mu_{ijk} = 0 \tag{41}$$

satisfy the stress equations of equilibrium identically. In addition, from (6),

$$\mu_{ijk} = 0 \quad \text{implies} \quad \kappa_{ijk} = 0, \tag{42}$$

$$\sigma_{ij} = 0 \quad \text{implies} \quad \sigma_{(ij)} = 0, \qquad \sigma_{[ij]} = 0, \tag{43}$$

where

$$\sigma_{(ij)} = \tfrac{1}{2}(\sigma_{ij}+\sigma_{ji}), \qquad \sigma_{[ij]} = \tfrac{1}{2}(\sigma_{ij}-\sigma_{ji}).$$

However,

$$\sigma_{[ij]} = (b_2-b_3)\gamma_{[ij]}, \tag{44}$$

$$\sigma_{(ij)} = g_1\delta_{ij}\varepsilon_{kk}+2g_2\varepsilon_{ij}+b_1\delta_{ij}\gamma_{kk}+(b_2+b_3)\gamma_{(ij)}. \tag{45}$$

Thus, $\sigma_{[ij]} = 0$ implies $\gamma_{[ij]} = 0$, and, solving (45) for $\gamma_{(ij)}$ in terms of ε_{ij} with $\sigma_{(ij)} = 0$, we find

$$\gamma_{(ij)} = -\frac{1}{(b_2+b_3)}\left[g_1-\frac{b_1(3g_1+2g_2)}{3b_1+b_2+b_3}\right]\delta_{ij}\varepsilon_{kk}-\left[\frac{2g_2}{b_2+b_3}\right]\varepsilon_{ij}, \tag{46}$$

whence

$$\gamma_{ii} = -\left[\frac{3g_1+2g_2}{3b_1+b_2+b_3}\right]\varepsilon_{ii}. \tag{47}$$

Inserting (46) and (47) into (6a) and noting that $\gamma_{[ij]} = 0$, we obtain

$$\tau_{ij} = \tilde{\lambda}\delta_{ij}\varepsilon_{kk} + 2\tilde{\mu}\varepsilon_{ij} \tag{48}$$

in analogy to the classical theory of elasticity. The form (48) was also obtained by Mindlin [1] in connection with a low frequency long wave-length approximation to his micro-structure theory.

Equations (48) can be inverted to give the strains in terms of the stresses, and it is evident from (46), (47) and (48) that the assumptions (41) will lead to solutions with

$$\varepsilon_{ij} = \text{constant}, \qquad \gamma_{ij} = \text{constant}, \qquad \kappa_{ijk} = 0.$$

Solutions of this form will satisfy Mindlin's compatibility equations, and hence can be integrated to give u_i and ψ_{ij}.

Uniform tension

A prism (with axis in the x_1 direction in rectangular cartesian coordinates) which is subjected to a uniform tension, T, over its plane ends and which is free from traction on its lateral surfaces is in equilibrium under the stress field

$$\tau_{11} = T, \qquad \text{all other} \quad \tau_{ij} = 0, \qquad \sigma_{ij} = 0, \qquad \mu_{ijk} = 0.$$

Employing (48) we find

$$\varepsilon_{11} = \frac{(\tilde{\lambda}+\tilde{\mu})T}{\tilde{\mu}(3\tilde{\lambda}+2\tilde{\mu})}, \qquad \varepsilon_{22} = \varepsilon_{33} = \frac{-\tilde{\lambda}T}{2\tilde{\mu}(3\tilde{\lambda}+2\tilde{\mu})}.$$

As in classical elasticity (see for example [4]) we may now define a Young's modulus, $\tilde{E}$, and a Poisson's ratio, $\tilde{\nu}$, by

$$\tilde{E} \equiv \frac{T}{\varepsilon_{11}} = \frac{\tilde{\mu}(3\tilde{\lambda}+2\tilde{\mu})}{\tilde{\lambda}+\tilde{\mu}}, \qquad \tilde{\nu} \equiv -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\tilde{\lambda}}{2(\tilde{\lambda}+\tilde{\mu})},$$

and the numerical values that were assigned to E and ν in the classical theory will now be assigned to $\tilde{E}$ and $\tilde{\nu}$. Consequently, $\tilde{\lambda}$ and $\tilde{\mu}$ will be assigned the numerical values formerly assigned to the Lamé constants λ and μ of the classical theory. Now, employing the first and the second of the inequalities (12) and the definition of $\tilde{\nu}$ we obtain

$$\frac{3\tilde{\lambda}+2\tilde{\mu}}{2\tilde{\mu}} = \frac{1+\tilde{\nu}}{1-2\tilde{\nu}} > 0,$$

and hence $-1 < \tilde{\nu} < \frac{1}{2}$ for positive definiteness. If we now restrict our consideration to "normal" materials, by which we shall mean materials for which quantities analogous to Poisson's ratio are positive and for which the micro-deformation has the same sign as the macro-deformation, then we may delimit the range of $\nu_5(\equiv \tilde{\nu})$ by $0 \leq \nu_5 < \frac{1}{2}$.

Hydrostatic pressure

A body of arbitrary shape subjected to a hydrostatic pressure, P, will be in equilibrium under the stress field

$$\tau_{11} = \tau_{22} = \tau_{33} = -P, \qquad \tau_{ij} = 0 \quad (i \neq j), \qquad \sigma_{ij} = 0, \qquad \mu_{ijk} = 0.$$

The macro-dilatation, ε_{ii}, is equal to $-3P/(3\tilde{\lambda}+2\tilde{\mu})$, and γ_{ii} is given by (47). Thus,

$$\nu_2(\equiv (3g_1+2g_2)/(3b_1+b_2+b_3))$$

appears as the negative of the ratio of the relative dilatation to the macro-dilatation. We have the intuitive idea that, for normal materials under hydrostatic pressure, the micro-dilatation, ψ_{ii}, will be somewhat less than the macro-dilatation and hence that the relative dilatation, γ_{ii}, will have the same sign as ε_{ii}. This implies that $\nu_2 \leq 0$. Furthermore, from (47), it is evident that $\nu_2 = 0$ corresponds to a limiting case in which the material is micro-homogeneous in bulk, i.e., the micro-dilatation coincides with the macro-dilatation, and that $\nu_2 = -1$ corresponds to a limiting case in which the micro-medium is incompressible. Thus, the appropriate range for ν_2 is $-1 \leq \nu_2 \leq 0$.

Shear

A cube subjected to a uniform shear, T, on its faces normal to the x_1, x_2 directions is in equilibrium under the stress field

$$\tau_{12} = \tau_{21} = T, \qquad \text{all other} \quad \tau_{ij} = 0, \qquad \sigma_{ij} = 0, \qquad \mu_{ijk} = 0.$$

Employing (48) we find that $\varepsilon_{12} = \varepsilon_{21} = T/2\tilde{\mu}$ and employing (46) we find that

$$\gamma_{(12)} = -\left[\frac{2g_2}{b_2+b_3}\right]\varepsilon_{12} = -\nu_1\varepsilon_{12}.$$

Following similar arguments to those used in establishing the range for ν_2, we conclude that the appropriate range for ν_1 is $-1 \leq \nu_1 \leq 0$.

Results from wave propagation and the constitutive equations

From the studies of plane wave propagation in [1] we see that ν_3 can be determined from the relation

$$\nu_3 \equiv \frac{\tilde{\lambda}+2\tilde{\mu}}{b_2+b_3} = \frac{3\rho\tilde{v}_1^2}{\rho' d^2\omega_s^2},$$

where d is half the length of a unit cell (assumed to be a cube), ρ and ρ' are the total mass per unit macro-volume and the mass of micro-material per unit macro-volume respectively, $\tilde{v}_1$ is the limiting group velocity of longitudinal waves at zero frequency and zero wave number and ω_s is the cut-off frequency for longitudinal-optical waves. An estimate for ν_3 can be obtained from the results of recent studies of bismuth ([5], [6], [7]). However, since bismuth is trigonal and not isotropic we must allow for some variation in the computed ν_3. Thus, taking

$$\tilde{v}_1 = 2\times 10^5 \text{ cm/sec}, \qquad \omega_s = 1{\cdot}5\times 10^{13} \text{ rad/sec}, \qquad d = 3 \text{ Å},$$

and assuming $\rho = \rho'$, we calculate $\nu_3 \approx 0{\cdot}6$, which we shall take as a typical value. In any case, we are assured from the conditions for positive definiteness, (12), that $\nu_3 > 0$.

Again, from the studies of plane wave propagation in [1], we see that ν_4 can be determined from the relation

$$\nu_4 \equiv \frac{\tilde{\lambda}+2\tilde{\mu}}{3b_1+b_2+b_3} = \frac{3\rho\tilde{v}_1^2}{\rho' d^2\omega_d^2},$$

where ρ, ρ', d and $\tilde{v}_1$ were defined previously, and ω_d is the cut-off frequency for longitudinal-dilatational waves. To the writer's knowledge, no experimental values of ω_d have been observed and hence we must at present resort to intuitive arguments in order to establish an appropriate range of values of ν_4. We note that

$$\frac{\nu_4}{\nu_3} = \frac{b_2+b_3}{3b_1+b_2+b_3},$$

and, from the form of the constitutive equations, we can interpret b_2+b_3 as a relative stiffness in shear and $3b_1+b_2+b_3$ as a relative stiffness in dilatation. It is plausible to expect that the ratio of these two stiffnesses is analogous to the ratio of the corresponding stiffnesses, 2μ and $3\lambda+2\mu$, in the classical theory of elasticity leading, for a normal material, to the range $0 < \nu_4/\nu_3 \leq 1$.

For the determination of appropriate ranges of values of ν_6 and ν_7 we employ the inequalities (see (12))

$$\nu_6 > 0, \qquad \nu_6 - 2\nu_7^2 > 0, \tag{49}$$

and an analogy with the classical theory of elasticity. Suppose that at some point in a body the only non-zero component of the double stress is μ_{111} (in rectangular cartesian coordinates x_1, x_2, x_3). An examination of the constitutive equations (6c) reveals that this double stress will be accompanied by non-zero micro-deformation gradient components κ_{111}, κ_{122}, κ_{133}, κ_{212}, κ_{313}, κ_{221}, and κ_{331}. Suppose now that it is desired to eliminate all the κ_{ijk} except for κ_{111}. It is clear from the constitutive equations that to do this requires the application of the double stresses μ_{122}, μ_{133}, μ_{212}, μ_{313}, μ_{221}, and μ_{331}. In particular, $\mu_{122}(= \xi_2\kappa_{111})$ must be applied to eliminate the micro-Poisson-like effect, κ_{122}, produced primarily by μ_{111}. Also, it is clear from Fig. 2 in [1] that, in a normal material, for $\kappa_{111} > 0$ we must have $\mu_{122} \geq 0$, and hence $\xi_2 \geq 0$. Furthermore, if we argue that applying a μ_{122} to eliminate the κ_{122} caused primarily by a μ_{111} in the micro-structure theory is analogous to applying a τ_{22} to eliminate the ε_{22} caused primarily by a τ_{11} in the classical theory, then for normal materials we will have

$$0 \leq \frac{\mu_{122}}{\mu_{111}} = \frac{\xi_2\kappa_{111}}{\xi_1\kappa_{111}} = \nu_7 < 1.$$

Once a value of ν_7 is chosen, a range of values of ν_6 can be selected which satisfies the inequalities (49).

For the determination of an appropriate range of values of ν_8 we note that

$$\xi_1 \equiv (a_1+a_2+a_3)+(a_2+a_5+a_8)+\xi_2+\xi_4$$

and hence, according to the sign of $[(a_1+a_2+a_3)+(a_2+a_5+a_8)+\xi_2]$, ξ_1 can be greater than or less than ξ_4. However, it seems reasonable to suppose that ξ_1 and ξ_4 are of the same order of magnitude for a typical material, and noting that (12) requires that $\nu_8 > 0$ for positive definiteness, we take $1/10 \leq \nu_8 \leq 10$ as a comprehensive range of values of ν_8.

In order to nondimensionalize the solution, it is necessary to single out a material property with the dimension of length. It is clear from (5) and from the definitions of ε_{ij}, γ_{ij}, and κ_{ijk}, that any ratio of a linear combination of the a_i's to a linear combination of λ, μ, the b_i's and the g_i's or any linear combination of such ratios would suffice. As more becomes known about the elastic constants of materials with micro-structure it may be possible to relate a ratio constructed in this way to some characteristic dimension of the micro-structure. For the present, it is convenient to choose as fundamental one of the lengths λ_1, λ_2 which appear explicitly in the solution; we choose the larger of the two, λ_1.

It is evident from a comparison of equation (21) with the corresponding governing equation of the classical theory, that much of the difference between the results of the two theories can be measured in terms of the parameters λ_1 and λ_2. If these lengths are small compared to the radius of the cavity, then the effects of the micro-structure will also be small. It is also apparent from a comparison of equations (3) of [3] with equations (28) and (29) of [3] that a similar statement (concerning the effect on the classical solution of the relative magnitude of length parameters of the material to a characteristic dimension of the body under consideration) can be made for more general boundary-value problems. In view of this, and the fact that the predictions of the classical theory of elasticity have been substantiated by photoelastic experiments on bodies of macroscopic dimensions and by vibration experiments at wavelengths of the order of a few thousand angstroms, it appears that quantities like λ_1 and λ_2 are certainly very small compared to macroscopic dimensions and may indeed be of the order of magnitude of a dimension of the micro-structure. In fact, a calculation* of the magnitude of a quantity, analogous to λ_1 and λ_2, based on data obtained by Germer, MacRae and Hartman [8] in low-energy electron diffraction experiments, shows that this quantity is approximately 5/8 of the interplanar distance of atoms. On the other hand, for the boundary-value problem under consideration, the radius of the cavity should probably be several times larger than a typical dimension of a crystal lattice if the continuum micro-structure theory is to be applicable. Thus, the ratio a/λ_1 should be greater than unity, and we shall take this as the lower limit in the numerical calculations.

It is evident that it is much more difficult to choose a small range of appropriate values for the ratio a/λ_1 than it is for the ν_1, and the numerical calculations are designed with this in mind. In order to keep the numerical work to a minimum while still exhibiting the characteristic features of the problem at hand it is convenient to choose a hypothetical 'standard' material (by assigning a set of values to the ratios ν_1) and to observe the behavior of the solution as the ratio a/λ_1 is varied. In addition, departures from the standard are also considered in order to show the effects of varying the ν_i.

RESULTS

On the basis of the remarks in the preceding section a standard set of values of the ν_i is chosen to be

$$\begin{aligned} \nu_1 &= -0{\cdot}10, & \nu_5 &= 0{\cdot}30, \\ \nu_2 &= -0{\cdot}10, & \nu_6 &= 1{\cdot}00, \\ \nu_3 &= 0{\cdot}60, & \nu_7 &= 0{\cdot}30, \\ \nu_4 &= 0{\cdot}20, & \nu_8 &= 1{\cdot}00, \end{aligned}$$

and computations are performed to observe how the solution changes when one of the ν_i is allowed to vary from its standard value while the other ν_i are either fixed at their standard values or restrained to remain in their fixed standard relationship to the varying ν_i. The results of these computations are exhibited graphically in Fig. 1.

Two interesting characteristics of the solution are revealed. The first is that, in contrast to the solution obtained by employing the classical theory of elasticity, where the stress

* R. D. Mindlin (private communication).

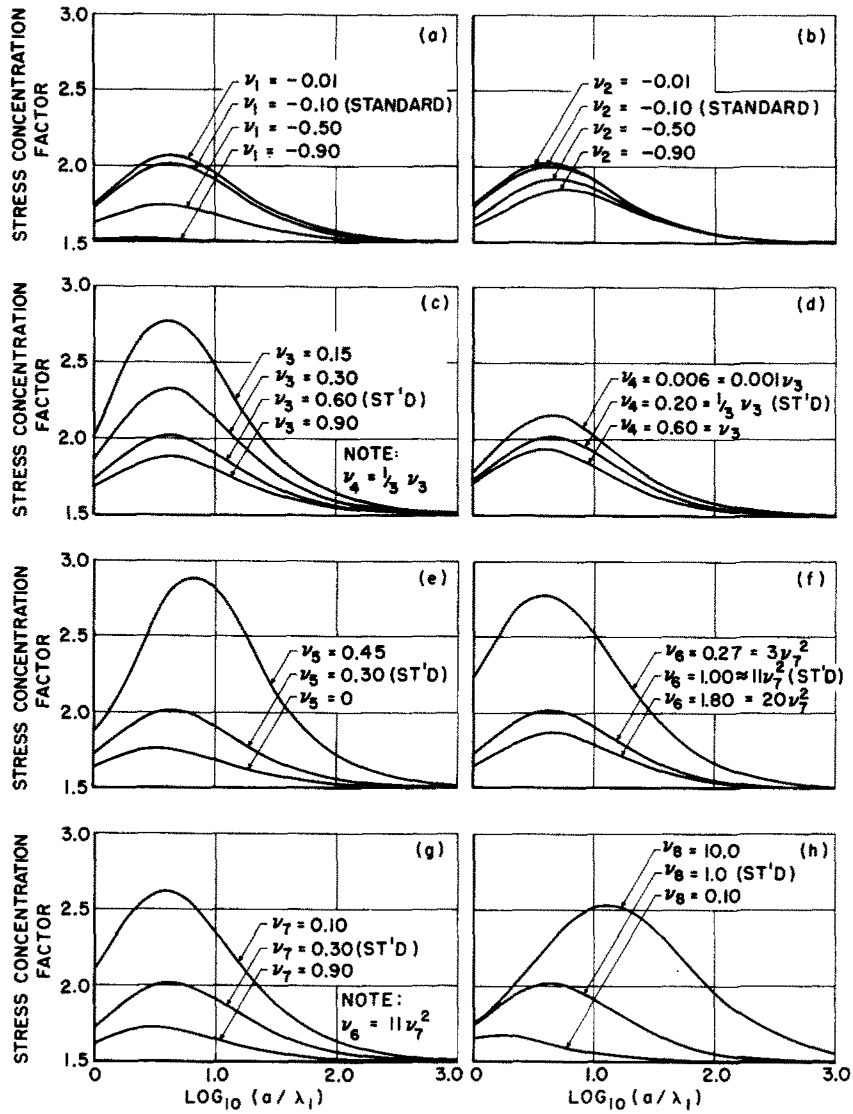

FIG. 1. Stress concentration factors for various hypothetical materials.

concentration factor is a constant, 3/2, independent of material properties and the radius of the cavity, the solution obtained by employing the theory of an elastic material with micro-structure shows a stress concentration factor which depends on both material properties and radius. In all the cases considered, the stress concentration factor is higher than the classical value of 3/2. However, as the ratio (a/λ_1) approaches infinity, the micro-structure solution approaches the classical solution and, on the scale of Fig. 1, the difference between these solutions is indistinguishable for $(a/\lambda_1) > 10^3$.

The second interesting characteristic is the appearance of a maximum stress concentration factor at an intermediate value of (a/λ_1). In view of the previously mentioned

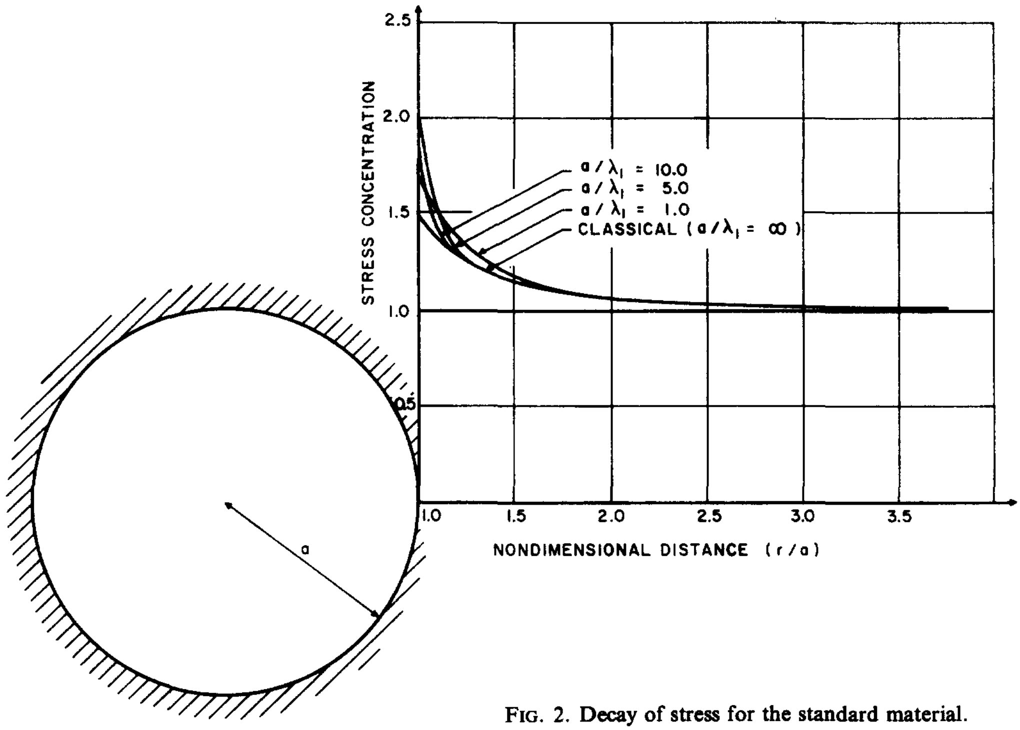

FIG. 2. Decay of stress for the standard material.

experiments of Germer, MacRae, and Hartman, this critical value probably corresponds to cavities with diameters one to ten times as large as the interplanar distance of atoms.

In Fig. 2 the stress ratio, $(\tau_{\theta\theta}+\sigma_{\theta\theta})/t_0$, is plotted as a function of a nondimensional measure, r/a, of the distance r, from the center of the cavity, for the standard material with several different values of a/λ_1. It is apparent from this graph that the difference between the micro-structure solution and the classical solution is localized at the surface of the cavity. For values of $r/a > 2$, the solutions are indistinguishable on the present scale.

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APPENDIX

On positive definiteness

In this section of the appendix are outlined the steps involved in determining a set of necessary and sufficient conditions for positive definiteness of the potential energy density.

From the discussion in the second section of this paper it is clear that if a quadratic form is written in matrix notation $q = \mathbf{XMX}^T$ where $\mathbf{M}$ is a real, symmetric, nxn matrix (with elements m_{ij}) then a set of necessary and sufficient conditions for q to be positive definite is given by:

$$m_{11} > 0, \begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} m_{11} \cdots m_{1n} \\ \cdots \\ m_{n1} \cdots m_{nn} \end{vmatrix} > 0.$$

When these conditions are imposed on the matrix $\mathbf{A}$ of the quadratic form (8) it is evident that each of the submatrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ can be treated separately. We obtain from $\mathbf{A}_1$:

$$\begin{aligned} b_1+b_2+b_3 &> 0, \\ (b_2+b_3)(2b_1+b_2+b_3) &> 0, \\ (b_2+b_3)^2(3b_1+b_2+b_3) &> 0, \\ (b_2+b_3)^2(3b_1+b_2+b_3)(\tilde{\lambda}+2\tilde{\mu}) &> 0, \\ 4(b_2+b_3)^2(3b_1+b_2+b_3)\tilde{\mu}(\tilde{\lambda}+\tilde{\mu}) &> 0, \\ 4(b_2+b_3)^2(3b_1+b_2+b_3)\tilde{\mu}^2(3\tilde{\lambda}+2\tilde{\mu}) &> 0; \end{aligned}$$

from $\mathbf{A}_2$:

$$\begin{aligned} b_2 &> 0, \\ (b_2+b_3)(b_2-b_3) &> 0, \\ (b_2+b_3)(b_2-b_3)\tilde{\mu} &> 0; \end{aligned}$$

from $\mathbf{A}_3$:

$$\begin{aligned} \xi_1 &> 0, \\ \xi_1(a_4+a_{10}+a_{13})-\xi_2^2 &> 0, \\ (a_{10}+a_{13})\Delta_1 &> 0, \\ \tfrac{1}{2}\Delta_1\Delta_2+(a_{10}+a_{13})\Delta_4 &> 0, \\ 2\Delta_2\Delta_4 &> 0, \\ \xi_4\Delta_3\Delta_4+\Delta_2\Delta_5 &> 0, \\ 2\xi_4\Delta_3\Delta_5 &> 0; \end{aligned} \tag{50}$$

and from $\mathbf{A}_4$:

$$\begin{aligned} a_{10} &> 0, \\ (a_{10}+a_{11})(a_{10}-a_{11}) &> 0, \\ (a_{10}-a_{11})^2(a_{10}+2a_{11}) &> 0, \\ \tfrac{1}{3}(a_{10}-a_{11})[\xi_4\xi_5(a_{10}-a_{11})+2(a_{10}+2a_{11})\Delta_3] &> 0, \\ \tfrac{1}{3}\Delta_3[2\xi_4\xi_5(a_{10}-a_{11})+(a_{10}+2a_{11})\Delta_3] &> 0, \\ \xi_4\xi_5\Delta_3^2 &> 0, \end{aligned}$$

where $\tilde{\lambda}$, $\tilde{\mu}$, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, Δ_1, Δ_2, Δ_3, Δ_4, and Δ_5 are defined in (11) and (12). It can now be shown that the inequalities (12) are necessary and sufficient for the inequalities (50) and hence necessary and sufficient for the potential energy density to be positive definite.

Proof of inequalities (23)

Proof that $a > 0$: It can be shown from (12) and (16) that

$$\begin{aligned} k_{11}-k_{12}^2/k'_{22}-k_{13}^2/k'_{33} &= \tilde{\lambda}+2\tilde{\mu} > 0, \\ k'_{22} &= \tfrac{3}{2}(b_2+b_3) > 0, \\ k'_{33} &= 3(b_1+b_2+b_3) > 0, \end{aligned} \tag{51}$$

and hence $k_{11} > 0$. Also,

$$k_{22}k_{33}-k_{23}^2 = \tfrac{9}{2}(\xi_1\xi_3-2\xi_2^2) > 0; \tag{52}$$

thus

$$a = k_{11}(k_{22}k_{33}-k_{23}^2) > 0.$$

Proof that $c > 0$:

$$c = k'_{22}k'_{33}(k_{11}-k_{12}^2/k'_{22}-k_{13}^2/k'_{33}),$$

and employing (51) we see that $c > 0$.

Proof that $b > 0$: We define

$$\begin{aligned} \phi_1 &= k_{11}(k'_{22}+k'_{33})+2k_{12}k_{13}-k_{12}^2-k_{13}^2, \\ \phi_2 &= \tfrac{1}{2}[k_{11}(2k'_{22}-k'_{33})+k_{12}k_{13}-2k_{12}^2+k_{13}^2], \\ \phi_3 &= k_{11}(2k'_{22}+\tfrac{1}{2}k'_{33})-2k_{12}k_{13}-2k_{12}^2-\tfrac{1}{2}k_{13}^2. \end{aligned}$$

Then,

$$b = \xi_1\phi_1+4\xi_2\phi_2+\xi_3\phi_3,$$

and

$$\phi_1\phi_3-2\phi_2^2 = \tfrac{9}{2}k_{11}c > 0. \tag{53}$$

It can be shown that $\phi_3 > 0$ is a necessary condition for positive definiteness of the potential energy density, and hence from (53) it is clear that $\phi_1 > 0$. Likewise, it is evident from (12) that $\xi_1 > 0$ and $\xi_3 > 0$. It is convenient to define

$$\alpha^2 \equiv 2\xi_2^2/\xi_1\xi_3, \qquad \beta^2 \equiv 2\phi_2^2/\phi_1\phi_3,$$

whence (52) implies $0 \leq \alpha < 1$, and (53) implies $0 \leq \beta < 1$. The relations

$$b = \xi_1\phi_1 + 4\xi_2\phi_2 + \xi_3\phi_3, \tag{54}$$

$$\xi_1 > 0, \qquad \xi_3 > 0, \qquad \xi_2^2 = \alpha^2\xi_1\xi_3, \qquad 0 \leq \alpha < 1, \tag{55a}$$

$$\phi_1 > 0, \qquad \phi_3 > 0, \qquad \phi_2^2 = \beta^2\phi_1\phi_3, \qquad 0 \leq \beta < 1, \tag{55b}$$

are sufficient for proving $b > 0$. Suppose that ξ_2 and ϕ_2 have the same sign; then $b > 0$ by inspection since it is a sum of positive terms. Suppose, on the other hand, that ξ_2 and ϕ_2 have opposite signs; then from (54),

$$b = \xi_1\phi_1 - 4|\xi_2\phi_2| + \xi_3\phi_3.$$

However, from the relations (55),

$$4\xi_2^2\phi_2^2 = \alpha^2\beta^2\xi_1\phi_1\xi_3\phi_3,$$

thus,

$$2|\xi_2\phi_2| = \alpha\beta(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}},$$

where we take the positive sign for the square root. Thus, employing (55),

$$\begin{aligned} b &= \xi_1\phi_1 - 2\alpha\beta(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}} + \xi_3\phi_3, \\ b &> \xi_1\phi_1 - 2(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}} + \xi_3\phi_3, \\ b &> [(\xi_1\phi_1)^{\frac{1}{2}} - (\xi_3\phi_3)^{\frac{1}{2}}]^2, \\ b &> 0. \end{aligned}$$

Proof that $b^2 - 4ac \geq 0$: The relations (55), and the identity

$$b^2 - 4ac = (\xi_1\phi_1 + 4\xi_2\phi_2 + \xi_3\phi_3)^2 - 4(\xi_1\xi_3 - 2\xi_2^2)(\phi_1\phi_3 - 2\phi_2^2) \tag{56}$$

are sufficient for proving that $b^2 - 4ac \geq 0$. We can write (56) in the form

$$b^2 - 4ac = (\xi_1\phi_1 - \xi_3\phi_3)^2 + 8\xi_2\phi_2(\xi_1\phi_1 + \xi_3\phi_3) + 8\xi_1\xi_3\phi_2^2 + 8\phi_1\phi_3\xi_2^2. \tag{57}$$

Suppose that ξ_2 and ϕ_2 have the same sign; then $b^2 - 4ac \geq 0$ by inspection since it is a sum of positive terms. Suppose, on the other hand, that ξ_2 and ϕ_2 have opposite signs; then, as in the previous proof,

$$b^2 - 4ac = (\xi_1\phi_1 - \xi_3\phi_3)^2 - 4\alpha\beta(\xi_1\phi_1 + \xi_3\phi_3)(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}} + 4(\alpha^2 + \beta^2)\xi_1\phi_1\xi_3\phi_3. \tag{58}$$

Making use of the inequality,

$$\delta_1^2 + \delta_2^2 \geq 2\delta_1\delta_2, \tag{59}$$

which holds for any real δ_1 and δ_2, it follows from (58) that

$$b^2 - 4ac \geq (\xi_1\phi_1 - \xi_3\phi_3)^2 - 4\alpha\beta[(\xi_1\phi_1 + \xi_3\phi_3)(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}} - 2\xi_1\phi_1\xi_3\phi_3]. \tag{60}$$

If in (59) we insert $\delta_1 = (\xi_1\phi_1)^{\frac{1}{2}}$ and $\delta_2 = (\xi_3\phi_3)^{\frac{1}{2}}$, we obtain

$$\xi_1\phi_1 + \xi_3\phi_3 \geq 2(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}},$$

and hence

$$(\xi_1\phi_1+\xi_3\phi_3)(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}} \geq 2\xi_1\phi_1\xi_3\phi_3.$$

Thus, from (60),

$$b^2-4ac \geq (\xi_1\phi_1-\xi_3\phi_3)^2-4[(\xi_1\phi_1+\xi_3\phi_3)(\xi_1\phi_1\xi_3\phi_3)^{\frac{1}{2}}-2\xi_1\phi_1\xi_3\phi_3],$$

$$b^2-4ac \geq [(\xi_1\phi_1)^{\frac{1}{2}}-(\xi_3\phi_3)^{\frac{1}{2}}]^4,$$

$$b^2-4ac \geq 0.$$

The equality sign holds if and only if $\xi_1\phi_1 = \xi_3\phi_3$ and $\alpha = \beta = 0$ which appears to be possible without violating positive definiteness of the potential energy density.

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Résumé—Dans cette étude, le problème de valeur-limite de la concentration d'effort à une cavité sphérique, dans un champs de tension isotropique, est résolu dans le cadre de la théorie de Mindlin sur un continuum élastique ayant une micro-structure déformable. On a constaté que le facteur de la concentration d'effort est légèrement supérieur aux 3/2 de l'élasticité classique pour une grande gamme de propriétés matérielles, et que les rapports du rayon de la cavité à un paramètre de longueur de la matière—avec un rapport critique, sont presque indépendants des propriétés matérielles restantes pour lesquelles le facteur de la concentration d'effort est maximum.

Zusammenfassung—In dieser Abhandlung wird das Problem des Grenzwertes für Spannungsüberhöhung an einem kugelförmigen Hohlraum in einem Gebiet isotropischer Spannung im Rahmen der Mindlinschen Theorie eines elastischen Kontinuums mit einer formveränderungsfähigen Mikro-Struktur gelöst. Es wurde festgestellt, dass der Spannungskonzentrationsfaktor ein im mässigen Umfang grösserer ist als 3/2 der klassischen Elastizität für einen weiten Bereich von Material Eigenschaften und Verhältnissen des Hohlraum Halbmessers zum Längen-Parameter des Materials, mit einem kritischen Verhältnis fast unabhängig von den verbleibenden Eigenschaften des Materials, für welche der Spannungskonzentrations Faktor ein Maximum ist.

Абстракт—Задача концентрации напряжения на сферической полости в поле изотропного напряжения решена в рамках теории Миндлина о эластичном континууме с деформирующейся микроструктурой. Установленно, что фактор концентрации напражения несколько выше 3/2 классической эластичности для большого разнообразия свойств материала и коэффициентов радиуса полости по отношению к параметрам длины материала, с критическим соотношением (почти независимым от остальных свойств материала) при котором показатель концентрации напряжения имеет максимальную величину.